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Some Applications of Jordan Norms to Involutorial Simple Associative Algebras

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The following result on reduced norms of central simple associative algebras has been proved by the author:

THEOREM. *Let $\mathscr{A}$ and $\mathscr{B}$ be central simple associative algebras of the same degree. Then the reduced norm forms on $\mathscr{A}$ and $\mathscr{B}$ are equivalent if and only if $\mathscr{A}$ and $\mathscr{B}$ are either isomorphic or anti-isomorphic.*

This was first proved by the author in [6] with the restriction that char $\Phi \neq 2, 3$ for Φ the base field. The restriction char $\Phi \neq 3$ and char $\Phi \neq 2$ were removed in [7] and [9], respectively (see also [16]). We also have the obvious result that $\mathscr{A}$ is a division algebra if and only if the reduced norm is anisotropic ($N(a) \neq 0$ if $a \neq 0$).

These results apply in particular to central simple algebras that are involutorial. However, as we shall show in this paper, these results can be improved by replacing N by a norm N_J of degree n, the degree of $\mathscr{A}$, in $n(n+1)/2$ variables, and if n is even by a form N_J of degree $n/2$ in $n(n-1)/2$ variables. The special case in which $n=2$ gives well-known results on quaternion algebras and the case $n=4$ gives an improvement of a result of Albert's and provides an answer to a question that had been raised (orally) by Professor Max Knus.

The reduced norm of a central simple algebra $\mathscr{A}$ defines a norm surface $N=0$ whose function field is a generic splitting field for $\mathscr{A}$ (Heuser [5], Saltman [13]). In a similar manner we can define a function field of the surface $N_J=0$. This is a generic splitting field for $\mathscr{A}$ if J is an involution of orthogonal type and is generic for fields E such that $\mathscr{A}^E \cong M_n(\mathscr{D})$, where $\mathscr{D}$ is a quaternion algebra if J is of symplectic type. The last section of the paper is devoted to the study of these fields.

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1. Forms Associated with an Involutorial Central Simple Associative Algebra

This note is an addendum to the author's paper [9], especially to Sections 7 and 8 of that paper. We follow the notations and freely use the results of [9]. Let $\mathscr{A}$ be a finite-dimensional central simple associative algebra over the field Φ such that $\{\mathscr{A}\}^2 = 1$ in the Brauer group $\mathrm{Br}(\Phi)$. It is a well-known result of Albert's that $\mathscr{A}$ has an involution J, that is, an anti-automorphism of $\mathscr{A}/\Phi$ such that $J^2 = 1_{\mathscr{A}}$ (Albert [2, p. 161]). The converse is clear since $\{\mathscr{A}\}^{-1} = \{\mathscr{A}^0\}$, $\mathscr{A}^0$, the opposite algebra of $\mathscr{A}$. The involution J defines the (unital quadratic) Jordan algebra $\mathscr{H}(\mathscr{A}, J)$. The basic Jordan composition here is $U_a b = aba$, $a, b \in \mathscr{H}(\mathscr{A}, J)$ and $\mathscr{H}(\mathscr{A}, J)$ contains 1. If $\bar{\Phi}$ is the algebraic closure of Φ, then the extension algebra $\mathscr{A}^{\bar{\Phi}} = \bar{\Phi} \otimes_{\Phi} \mathscr{A} = M_n(\bar{\Phi})$ and J extends to an involution $J^{\bar{\Phi}}$ in $M_n(\bar{\Phi})$. Relative to a suitable choice of the matrix base $\{e_{ij} \mid 1 \leqslant i, j \leqslant n\}$, $J^{\bar{\Phi}}$ is either the transpose involution $a \rightsquigarrow {}^t a$ or the symplectic involution $a \rightsquigarrow s({}^t a) s^{-1}$, where

$$s = \mathrm{diag}\{q, q, \ldots, q\}, \qquad q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.1}$$

In the first case J is said to be of *orthogonal type* and in the second case J is of *symplectic type*. We have the following table of dimensionalities:

$$\begin{aligned} [\mathscr{H}(\mathscr{A}, J) : \Phi] &= n(n+1)/2 && \text{if } J \text{ is of orthogonal type,} \\ &= n(n-1)/2 && \text{if } J \text{ is of symplectic type} \\ & && \mathrm{char}\,\Phi \neq 2, \\ &= n(n+1)/2 && \text{if } J \text{ is of symplectic type and} \\ & && \mathrm{char}\,\Phi = 2, \end{aligned} \tag{1.2}$$

where n is the degree of $\mathscr{A}$ ($[\mathscr{A} : \Phi] = n^2$). In the case of char $\Phi = 2$ and J is of symplectic type it is generally more interesting to replace $\mathscr{H}(\mathscr{A}, J)$ by the Jordan algebra $\mathscr{H}(\mathscr{A}, J)'$ defined to be the outer ideal in $\mathscr{H}(\mathscr{A}, J)$ generated by 1. We have $\mathscr{H}(\mathscr{A}, J)'^{\bar{\Phi}} = \mathscr{H}(\mathscr{A}^{\bar{\Phi}}, J^{\bar{\Phi}})'$ and this is the set of matrices a such that as is alternate, that is ${}^t(as) = as$ and the diagonal elements of as are 0. We have

$$[\mathscr{H}(\mathscr{A}, J)' : \Phi] = n(n-1)/2. \tag{1.3}$$

In the remainder of the paper $\mathscr{H}(\mathscr{A}, J)'$ will denote the outer ideal generated by 1 in $\mathscr{H}(\mathscr{A}, J)$, J of symplectic type and char $\Phi = 2$.

The vector space $\mathscr{K}(\mathscr{A}, J)$ of J-skew elements of $\mathscr{A}$ is a Lie algebra with composition $[ab] = ab - ba$, which is restricted if char $\Phi = p \neq 0$. If char $\Phi \neq 2$, then $\mathscr{A} = \mathscr{H}(\mathscr{A}, J) \oplus \mathscr{K}(\mathscr{A}, J)$. If u is an invertible element in $\mathscr{H}(\mathscr{A}, J) \cup \mathscr{K}(\mathscr{A}, J)$, then $J_u \equiv i_u J$, where i_u is the inner automorphism $a \rightsquigarrow uau^{-1}$, is an involution in $\mathscr{A}$. It follows from the Skolem–Noether theorem that every involution in $\mathscr{A}$ is of this form. If $u \in \mathscr{H}(\mathscr{A}, J)$, then

$$\mathscr{H}(\mathscr{A}, J_u) = u\mathscr{H}(\mathscr{A}, J) \tag{1.4}$$

and the Jordan algebra $\mathscr{H}(\mathscr{A}, J_u)$ is isotopic to $\mathscr{H}(\mathscr{A}, J)$. In fact, the map $u_L : a \rightsquigarrow ua$ is an isomorphism of the u-isotope $\mathscr{H}(\mathscr{A}, J)^{(u)}$ of $\mathscr{H}(\mathscr{A}, J)$ onto $\mathscr{H}(\mathscr{A}, J_u)$. The same remarks apply to the Jordan algebra $\mathscr{H}(\mathscr{A}, J)'$ and any invertible element $u \in \mathscr{H}(\mathscr{A}, J)'$. If $u \in \mathscr{K}(\mathscr{A}, J)$ is invertible, then $\mathscr{H}(\mathscr{A}, J_u) = u\mathscr{K}(\mathscr{A}, J)$. Hence if char $\Phi \neq 2$, then J is of orthogonal type if and only if J_u is of symplectic type. If u_1 and u_2 are invertible elements of $\mathscr{K}(\mathscr{A}, J)$, then $\mathscr{H}(\mathscr{A}, J_{u_1})$ and $\mathscr{H}(\mathscr{A}, J_{u_2})$ are isotopic since $u_2 u_1^{-1} \in \mathscr{H}(\mathscr{A}, J_{u_1})$ and $(J_{u_1})_{u_2 u_1^{-1}} = J_{u_2}$.

We recall that a Jordan algebra $\mathscr{J}$ is a division algebra if every $a \neq 0$ in $\mathscr{J}$ is invertible ($\equiv U_a$ is invertible in $\mathrm{End}_\Phi \mathscr{J}$). If $\mathscr{A}$ is an associative division algebra then $\mathscr{H}(\mathscr{A}, J)$ is a Jordan division algebra.

1.5. PROPOSITION. *Assume the degree $n > 1$ (that is, $\mathscr{A} \neq \Phi$). Then $\mathscr{H}(\mathscr{A}, J)(\mathscr{H}(\mathscr{A}, J)')$ is a division algebra if and only if either $\mathscr{A}$ is a division algebra or $\mathscr{A} = M_2(\Phi)$ and J is of symplectic type.*

Proof. We may replace $(\mathscr{A}, J)$ by an isomorphic pair $(\mathscr{B}, K)$, where $(\mathscr{B}, K) \cong (\mathscr{A}, J)$ means that there exists an isomorphism η of $\mathscr{A}$ onto $\mathscr{B}$ such that $\eta J = K\eta$. We use the structure theorem that either $(\mathscr{A}, J) \cong (M_m(\mathscr{D}), \bar{t}_c)$, where $\mathscr{D}$ is a division algebra with an involution $d \rightsquigarrow \bar{d}$ and $\bar{t}_c$ is the involution $a \rightsquigarrow c({}^t\bar{a})c^{-1}$, $\bar{a} = (\bar{a}_{ij})$ for $a = (a_{ij})$ and $c = \mathrm{diag}\{c_1, c_2, \ldots, c_m\}$, where $c_i = \bar{c}_i$ is invertible, or $(\mathscr{A}, J) \cong (M_m(\mathscr{D}), t_s)$, where $\mathscr{D} = M_2(\Phi)$ and t_s is $a \rightsquigarrow s({}^t a)s^{-1}$, s as in (1.1) (see, for example, Jacobson [8, p. 0.11]). In either case $\mathscr{H}(\mathscr{A}, J)$ and $\mathscr{H}(\mathscr{A}, J)'$ contain m orthogonal idempotents $\neq 0$. Hence, if $\mathscr{H}(\mathscr{A}, J)$ and $\mathscr{H}(\mathscr{A}, J)'$ are division algebras, then $m = 1$. Then either $\mathscr{A}$ is a division algebra or $\mathscr{A} = M_2(\Phi)$ with J symplectic. Conversely, if $(\mathscr{A}, J)$ is of either of these forms, then $\mathscr{H}(\mathscr{A}, J)$ is a division algebra. ∎

From now on it is convenient to assume Φ infinite. Let N_J denote the generic norm defined on the Jordan algebra $\mathscr{H}(\mathscr{A}, J)$, and if char $\Phi = 2$ and J is of symplectic type, let N_J' denote the generic norm on $\mathscr{H}(\mathscr{A}, J)'$. For any characteristic, if J is of orthogonal type, then N_J is the restriction to $\mathscr{H}(\mathscr{A}, J)$ of the reduced (or generic) norm N on $\mathscr{A}$. Hence N_J is a form of degree $n = \deg \mathscr{A}$. If char $\Phi \neq 2$ and J is of symplectic type, then $\deg N_J = \frac{1}{2}n$. In this case, an exact determination of N_J is the following: If $\bar{\Phi}$ is the algebraic closure of Φ, then we have an imbedding of $\mathscr{A}$ in $M_n(\bar{\Phi})$ so that the elements $a \in \mathscr{H}(\mathscr{A}, J)$ satisfy the condition $s({}^t a)s^{-1} = a$. Then ${}^t(as) = -as$ and $N_J(a)$ is the Pfaffian $\mathrm{Pf}(as)$. If char $\Phi = 2$ and J is of symplectic type, then N_J is the restriction of the reduced norm to $\mathscr{H}(\mathscr{A}, J)$ and N_J' has degree $n/2$ and its determination is the same as that of N_J for char $\neq 2$ and J symplectic.

We can now determine the isotopy classes of Jordan algebras $\mathscr{H}(\mathscr{A}, J)$ and $\mathscr{H}(\mathscr{A}, J)'$. First, if $\deg \mathscr{A} = n$ is odd, then there are no symplectic involutions. Hence all the $\mathscr{H}(\mathscr{A}, J)$ are isotopic. Next, suppose n is even and char $\Phi \neq 2$. In this case there exist symplectic involutions. For, let J be of orthogonal type. Then the subset of $\mathscr{H}(\mathscr{A}, J)$ of invertible elements is given by the condition $N(u) \neq 0$. Hence this is a Zariski open subset of $\mathscr{H}(\mathscr{A}, J)$. Since the corresponding subset of $\mathscr{A}^{\bar{\Phi}}$ is not vacuous, there exist invertible u in $\mathscr{H}(\mathscr{A}, J)$ and we have a symplectic involution J_u. Thus if n is even and char $\Phi \neq 2$, then there exist two isotopy classes of algebras $\mathscr{H}(\mathscr{A}, J)$. The same argument applies in the case n even and char $\Phi = 2$. In addition, in this case we have the isotopy class of algebras $\mathscr{H}(\mathscr{A}, J)'$, J of symplectic type.

1.6. Definition. Let f and g be polynomial functions on the vector spaces V/Φ and W/Φ, respectively. Then f and g are *similar* ($f \sim g$) if there exists a bijective linear map η of V into W and a nonzero $\rho \in \Phi$ such that $g(\eta(x)) = \rho f(x)$, $x \in V$.

1.7. Proposition. *Let J and K be involutions of orthogonal (symplectic) type in $\mathscr{A}$. Then N_J and N_K (defined on $\mathscr{H}(\mathscr{A}, J)$ and $\mathscr{H}(\mathscr{A}, K)$, respectively) are similar. The same holds for N_J' and N_K' in the characteristic two case for J and K symplectic.*

Proof. If J and K are of orthogonal type, then $K = i_u J$ and $\mathscr{H}(\mathscr{A}, K) = u\mathscr{H}(\mathscr{A}, J)$ for an invertible element $u \in \mathscr{H}(\mathscr{A}, J)$. Then $\eta = u_L$ is a bijective linear map of $\mathscr{H}(\mathscr{A}, J)$ onto $\mathscr{H}(\mathscr{A}, K)$. Now N_J and N_K are the restrictions to $\mathscr{H}(\mathscr{A}, J)$ and $\mathscr{H}(\mathscr{A}, K)$, respectively, of the reduced norm N on $\mathscr{A}$ and this function is multiplicative. Hence $N_K(\eta(a)) = N(ua) = N(u)\,N(a) = \rho N(a)$ for $\rho = N(u)$. Thus $N_J \sim N_K$. Next let J and K be of symplectic type. Then $N_J(a)^2 = N(a)$, $N_K(b)^2 = N(b)$ for the reduced norm N. Again we have an invertible $u \in \mathscr{H}(\mathscr{A}, J)$ such that $K = i_u J$ and $\eta = u_L$ is a bijective linear map of $\mathscr{H}(\mathscr{A}, J)$ onto $\mathscr{H}(\mathscr{A}, K)$. We have $N_K(\eta(a))^2 = \rho N_J(a)^2$, $a \in \mathscr{H}(\mathscr{A}, J)$. Since $N_J(1) = 1$, this gives $\rho = \tau^2$, where $\tau = N_K(\eta(1))$. Then $N_K(\eta(a))^2 = \tau^2 N_J(a)$ and $N_K(\eta(a)) = \pm\tau N_J(a)$, $a \in \mathscr{H}(\mathscr{A}, J)$. Thus the polynomial function $a \rightsquigarrow [N_K(\eta(a)) + \tau N_J(a)][N_K(\eta(a) - \tau N_J(a)] = 0$. It follows that either $N_K(\eta(a)) = \tau N_J(a)$ or $N_K(\eta(a)) = -\tau N_J(a)$ for all $a \in \mathscr{H}(\mathscr{A}, J)$. Hence $N_K \sim N_J$. A similar argument applies to N_J' and N_K'. ∎

2. Norm Conditions for a Division Algebra and for Isomorphism

We recall that an element $a \in \mathscr{H}(\mathscr{A}, J)$ $(\mathscr{H}(\mathscr{A}, J)')$ is invertible if and only if $N_J(a) \neq 0$ $(N_J'(a) \neq 0)$. Hence $\mathscr{H}(\mathscr{A}, J)$ is a Jordan division algebra if and only if the form N_J (N_J') is *anisotropic* in the sense that $N_J(a) \neq 0$ $(N_J'(a) \neq 0)$ for every $a \neq 0$ in $\mathscr{H}(\mathscr{A}, J)$ $(\mathscr{H}(\mathscr{A}, J)')$. The following result is therefore an immediate consequence of Proposition 1.5.

2.1. Theorem. *Assume* $\deg \mathscr{A} > 1$. *Then* $N_J(N_{J'})$ *is anisotropic if and only if either* $\mathscr{A}$ *is a division algebra or* $\mathscr{A} = M_2(\Phi)$ *and* J *is of symplectic type.*

We recall the definition of the generic trace bilinear form in $\mathscr{H}(\mathscr{A}, J)$ as

$$T_J(a, b) = -\Delta_1^a \Delta^b \log N_J = (\Delta_1^a N_J)(\Delta_1^b N_J) - \Delta_1^a \Delta^b N_J, \tag{2.2}$$

where $\Delta_b^a F$ is the directional derivative at a in the direction b of the rational function F. We have a similar definition of $T_J'(a, b)$ defined by N_J'. Then T_J is a nondegenerate symmetric bilinear form if char $F \neq 2$, but not otherwise. However, T_J' is nondegenerate. From now on we discard the degenerate cases. We have the important formula

$$\Delta_c^a \Delta^b \log N_J = T_J(U_c^{-1} a, b) \tag{2.3}$$

for all a, b and invertible c and we have a similar formula for T_J'.

2.4. Theorem. *Let* $\mathscr{A}_i$, $i = 1, 2$, *be an involutorial central simple algebra of degree* $n > 1$, J_i *an involution in* $\mathscr{A}_i$ *of orthogonal* (*symplectic*) *type.*

(i) *If* char $\Phi \neq 2$, *we assume* $n > 2$ *in the symplectic case. Then* $\mathscr{A}_1 \cong \mathscr{A}_2$ *if and only if* $N_{J_1} \sim N_{J_2}$.

(ii) *If* char $\Phi = 2$, *we assume* $n > 2$ *and* J_i *is of symplectic type. Then* $\mathscr{A}_1 \cong \mathscr{A}_2$ *if and only if* $N_{J_1}' \sim N_{J_2}'$.

Proof. (i) The proof is similar to that of [5, Theorem 7, p. 244] (see also the proof of [7, Theorem 10]). Suppose $N_{J_1} \sim N_{J_2}$ and let η be a bijective linear map of $\mathscr{H}(\mathscr{A}_1, J_1)$ onto $\mathscr{H}(\mathscr{A}_2, J_2)$ and ρ a nonzero element of Φ such that

$$N_{J_2}(\eta(x)) = \rho N_{J_1}(x), \qquad x \in \mathscr{H}(\mathscr{A}_1, J_1). \tag{2.5}$$

Then x is invertible in $\mathscr{H}(\mathscr{A}_1, J_1)$ if and only if $\eta(x)$ is invertible in $\mathscr{H}(\mathscr{A}_2, J_2)$. From (2.3) and (2.5) we can obtain

$$T_{J_1}(U_c^{-1}b, a) = T_{J_2}(U_{\eta(c)}^{-1}\eta(b), \eta(a)) \tag{2.6}$$

for all a, b, and invertible c in $\mathscr{H}(\mathscr{A}_1, J_1)$. Since T_{J_1} and T_{J_2} are nondegenerate, we have a bijective linear map η^* of $\mathscr{H}(\mathscr{A}_2, J_2)$ onto $\mathscr{H}(\mathscr{A}_1, J_1)$ such that for $y_1 \in \mathscr{H}(\mathscr{A}_1, J_1)$, $x_2 \in \mathscr{H}(\mathscr{A}_2, J_2)$,

$$T_{J_1}(\eta^*(x_2), y_1) = T_{J_2}(x_2, \eta(y_1)). \tag{2.7}$$

Then by (2.6) and the nondegeneracy of T_{J_1} we obtain

$$U_{\eta(c)} = \eta U_c \eta^* \tag{2.8}$$

for all c with $N_{J_1}(c) \neq 0$. It follows that this holds for all c. This implies that η is an isomorphism of $\mathscr{H}(\mathscr{A}_1, J_1)$ onto an isotope of $\mathscr{H}(\mathscr{A}_2, J_2)$ [7, p. 116]. Now our hypotheses imply that $(\mathscr{A}_i, J_i)$ is a perfect algebra with involution. Hence $\mathscr{A}_1$ and the injection map constitute a special universal envelope for $\mathscr{H}(\mathscr{A}_1, J_1)$. By [7, Proposition 1, p. 119], $\mathscr{A}_2$ and a suitable map constitute a special universal envelope for the isotope of $\mathscr{H}(\mathscr{A}_2, J_2)$. The isomorphism of $\mathscr{H}(\mathscr{A}_1, J_1)$ with the isotope of $\mathscr{H}(\mathscr{A}_2, J_2)$ gives an isomorphism of $\mathscr{A}_1$ onto $\mathscr{A}_2$. Conversely, suppose $\mathscr{A}_1 \cong \mathscr{A}_2$ and let η be an isomorphism of $\mathscr{A}_1$ onto $\mathscr{A}_2$. Then $\eta J_1 \eta^{-1}$ is an involution in $\mathscr{A}_2$ of the same type as J_1 and η is a similarity of N_{J_1} and $N_{\eta J_1 \eta^{-1}}$. Thus $N_{J_1} \sim N_{\eta J_1 \eta^{-1}}$ and, by Proposition 1.7, $N_{\eta J_1 \eta^{-1}} \sim N_{J_2}$. Hence $N_{J_1} \sim N_{J_2}$.

(ii) The proof is identical to that of (i). ∎

3. Involutorial Central Simple Algebras of Degrees Two and Four

The central simple algebras of degree two are quaternion algebras. Any such algebra $\mathscr{A}$ has the *standard involution* J: $a \rightsquigarrow \bar{a}$ such that $a + \bar{a} = T(a)$, $a\bar{a} = N(a) = \bar{a}a$, where T and N are the reduced trace and norm on $\mathscr{A}$. The involution J is of symplectic type since $\mathscr{H}(\mathscr{A}, J) = \Phi$. Since $i_\alpha = 1$ if $\alpha \in \Phi$, J is the only involution of symplectic type in $\mathscr{A}$. Let $\mathscr{A}' = \{a \mid T(a) = 0\}$, so $\mathscr{A}'$ is three-dimensional and coincides with $\mathscr{K}(\mathscr{A}, J)$ $(=\mathscr{H}(\mathscr{A}, J))$ if char $\Phi = 2$). Let u be an invertible element of $\mathscr{K}(\mathscr{A}, J)$, $u \notin \Phi$. Then J_u is of orthogonal type and $\mathscr{H}(\mathscr{A}, J_u) = u\mathscr{A}'$. These remarks and Theorems 2.1 and 2.4 give the following well-known results:

3.1. Theorem. *Let $\mathscr{A}$ be a quaternion algebra over Φ, $\mathscr{A}'$ the subspace of $\mathscr{A}$ of elements of reduced trace* 0. *Then*

(i) $\mathscr{A}$ *is a division algebra if and only if* $N|\mathscr{A}'$ *is anisotropic.*

(ii) *If* char $\Phi \neq 2$ *and* $\mathscr{A}_i$, $i = 1, 2$, *is a quaternion algebra over* Φ, *then* $\mathscr{A}_1 \cong \mathscr{A}_2$ *if and only if* $N|\mathscr{A}_1' \sim N|\mathscr{A}_2'$.

Proof. (i) Choose $u \in \mathscr{A}'$, $u \notin \Phi$. Then J_u is of orthogonal type and $\mathscr{H}(\mathscr{A}, J_u) = u\mathscr{A}'$. Hence $N|\mathscr{A}' \sim N_{J_u}$ and $N|\mathscr{A}'$ is anisotropic if and only if N_{J_u} is anisotropic. The result now follows from Theorem 2.1.

(ii) This follows in the same way by replacing $N|\mathscr{A}_1'$ and $N|\mathscr{A}_2'$ by similar forms $N_{J_{u_1}}$ and $N_{J_{u_2}}$, where J_{u_i} is of orthogonal type, and then applying Theorem 2.4. ∎

Now let $\mathscr{A}$ be involutorial central simple of degree 4. By a theorem of Albert's, $\mathscr{A} = \mathscr{A}_1 \otimes_\Phi \mathscr{A}_2$, where $\mathscr{A}_i$ is a quaternion algebra (Albert [2] or Racine [10]). Let J_i be the standard involution in $\mathscr{A}_i$ and let u_1 be an element of $\mathscr{A}_1' = \mathscr{H}(\mathscr{A}_1, J_1)$, $u_1 \notin \Phi$. Then J_{1u_1} is of orthogonal type and hence $J = J_{1u_1} \otimes J_2$ is of symplectic type. If $a_i \in \mathscr{A}_i'$, then

$$l(a_1, a_2) = u_1 \otimes a_2 + u_1 a_2 \otimes 1 \in \mathscr{H}(\mathscr{A}, J) \tag{3.2}$$

and the set

$$\mathscr{L} = \{l(a_1, a_2) \mid a_i \in \mathscr{A}_i'\} \tag{3.3}$$

is a six-dimensional subspace of $\mathscr{H}(\mathscr{A}, J)$. Hence $\mathscr{L} = \mathscr{H}(\mathscr{A}, J)$ if char $\Phi \neq 2$. We shall see in a moment that $\mathscr{L} = \mathscr{H}(\mathscr{A}, J)'$ if char $\Phi = 2$.

Since a_1, u_1, and $a_1 + u_1 \in \mathscr{A}_1'$, $-N(u_1 + a_1) = (u_1 + a_1)^2 = u_1^2 + u_1 a_1 + a_1 u_1 + a_1^2 = -N(u_1) - N(u_2) + u_1 a_1 + a_1 u_1$, where N is the reduced norm in $\mathscr{A}_1$. Thus

$$u_1 a_1 + a_1 u_1 = N(u_1) + N(a_1) - N(u_1 + a_1).$$

On the other hand, a calculation with matrices of trace 0 shows that $N(u_1) + N(a_1) - N(u_1 + a_1) = T(u_1 a_1)$, T the reduced traces in $\mathscr{A}_1$. Hence

$$u_1 a_1 + a_1 u_1 = T(u_1 a_1). \tag{3.4}$$

Now we have

$$(u_1 a_1)^2 = T(u_1 a_1)\, u_1 a_1 - N(u_1 a_1). \tag{3.5}$$

Hence

$$\begin{aligned} l(a_1, a_2)^2 &= (u_1 \otimes a_2 + u_1 a_1 \otimes 1)^2 \\ &= N(u_1)\, N(a_2) + u_1^2 a_1 \otimes a_2 + u_1 a_1 u_1 \otimes a_2 + (u_1 a_1)^2 \otimes 1 \\ &= N(u_1)\, N(a_2) + T(u_1 a_1)\, u_1 \otimes a_2 + T(u_1 a_1)\, u_1 a_1 \otimes 1 \\ &\quad - N(u_1 a_1) \qquad ((3.4) \text{ and } (3.5)) \\ &= T(u_1 a_1)\, l(a_1, a_2) - N(u_1)(N(a_1) - N(a_2)). \end{aligned} \tag{3.6}$$

Now suppose char $\Phi = 2$. Then $\mathscr{H}(\mathscr{A}, J)'$ contains 1 and all squares of elements of $\mathscr{H}(\mathscr{A}, J)$. Then, by (3.6), $\mathscr{H}(\mathscr{A}, J)'$ contains every $l(a_1, a_2)$ with $T(u_1 a_1) \neq 0$. It follows that $\mathscr{H}(\mathscr{A}, J)'$ contains every $l(a_1, a_2)$. Hence $\mathscr{H}(\mathscr{A}, J)' \supset \mathscr{L}$ and since both are six-dimensional, $\mathscr{L} = \mathscr{H}(\mathscr{A}, J)'$ if char $\Phi = 2$. We have also shown that $l(a_1, a_2)$ is a root of

$$\lambda^2 - T(ua_1)\lambda + N(u_1)((N(a_1) - N(a_2)). \tag{3.7}$$

It follows that

$$N_J(l(a_1, a_2)) = N(u_1)((N(a_1) - N(a_2)) \tag{3.8}$$

if char $\Phi \neq 2$, where N_J is the generic norm in $\mathscr{H}(\mathscr{A}, J)$, and

$$N_J'(l(a_1, a_2)) = N(u_1)((N(a_1) - N(a_2)) \tag{3.9}$$

if char $\Phi = 2$, where N_J' is the generic norm in $\mathscr{H}(\mathscr{A}, J)'$.

3.10. Definition. Let $\mathscr{A}$ be involutorial central simple of degree 4 and $\mathscr{A} = \mathscr{A}_1 \otimes_\Phi \mathscr{A}_2$ a factorization of $\mathscr{A}$ as a tensor product of quaternion algebras. Then the quadratic form on $\mathscr{A}' \equiv \mathscr{A}_1' \oplus \mathscr{A}_2'$ defined by

$$Q(a_1 + a_2) = N(a_1) - N(a_2) \tag{3.11}$$

for $a_i \in \mathscr{A}_i'$ will be called an *Albert quadratic form* for $\mathscr{A}$.

Then we have

3.12. Theorem. (i) *Any two Albert quadratic forms for an involutorial central simple algebra of degree* 4 *are similar.*

(ii) *Two involutorial central simple algebras of degree* 4 *are isomorphic if and only if their Albert quadratic forms are similar.*

(iii). *If Q is an Albert quadratic form for $\mathscr{A}$, then the index of $\mathscr{A}$ is* 4, 2, *or* 1 *according as the Witt index of Q is* 0, 1, *or* 3.

Remarks. Part (iii) extends a result of Albert's in [1]. A similar result going in the direction of quadratic form to algebra of degree 4 via Clifford algebras has been given by Tamagawa in [12]. See also Seligman [11, p. 340]. Part (ii) gives an affirmative answer to a question which had been asked of the author by Professor Max Knus.

Proof. (i) We have shown that the Albert quadratic form Q associated with a factorization $\mathscr{A} = \mathscr{A}_1 \otimes \mathscr{A}_2$ is similar to a Jordan norm form N_J or N_J' where J is an involution of symplectic type in $\mathscr{A}$. Since the N_J and N_J' determined by two such involutions are similar, any two Albert quadratic forms are similar.

(ii) This is an immediate consequence of the argument in (i) and Theorem 2.4.

(iii) By Theorem 2.1, $\mathscr{A}$ is a division algebra if and only if N_J or N_J' is anisotropic. Since $N_J \sim Q$ or $N_J' \sim Q$, $\mathscr{A}$ is a division algebra if and only if Q is anisotropic. Thus $\mathscr{A}$ has index 4 if and only if the Witt index of Q is 0. Next, suppose $\mathscr{A}$ has index 1 so $\mathscr{A} \cong M_4(\Phi)$. Then $\mathscr{A} = \mathscr{A}_1 \otimes \mathscr{A}_2$, where $\mathscr{A}_1 \cong \mathscr{A}_2$. Then it is clear that the Witt index of Q is 3. Finally, suppose $\mathscr{A}$ has index 2. Then $\mathscr{A} = M_2(\Phi) \otimes \mathscr{D}$, where $\mathscr{D}$ is a division algebra. Then $\mathscr{A}'$ is an orthogonal direct sum of $M_2(\Phi)'$ and $\mathscr{D}'$ relative to Q. It is readily seen that the $Q \mid M_2(\Phi)'$ is of Witt index 1. Since $\mathscr{D}$ is a division algebra, $Q \mid \mathscr{D}'$ is anisotropic. It follows that Q has index 1. ∎

Theorem 3.12(iii) has the following consequence.

3.13 Corollary (Albert–Sah). *If $\mathscr{A}_i$, $i = 1, 2$, is a quaternion division algebra, then $\mathscr{A}_1 \otimes_\Phi \mathscr{A}_2$ is not a division algebra if and only if the $\mathscr{A}_i$ contain isomorphic quadratic subfields.*

Proof. The condition is sufficient that $\mathscr{A}_1 \otimes \mathscr{A}_2$ is not a division algebra since the tensor product of isomorphic extension fields $E_i/\Phi \neq \Phi$ is never a field. Now suppose $\mathscr{A}_1 \otimes \mathscr{A}_2$ is not a division algebra. Then the Albert quadratic form determined by this factorization is not anisotropic. Hence we have an element $a = a_1 + a_2 \neq 0$, $a_i \in \mathscr{A}_i'$ such that $N(a_1) = N(a_2)$. Since $N \mid \mathscr{A}_i'$ is anisotropic, $N(a_i) \neq 0$. Since $T(a_i) = 0$ and $a_i^2 - T(a_i)\, a_i + N(a_i) = 0$, we have $\Phi(a_1) \cong \Phi(a_2)$. ∎

4. Generic Norm Fields of an Involutorial Central Simple Algebra

The results we shall derive in this section are valid also for $\mathscr{H}(\mathscr{A}, J)'$, J symplectic, Φ of characteristic two. However, for the sake of simplicity of exposition we shall assume char $\Phi \neq 2$.

Let $(u_1, \ldots, u_m)$ be a base for $\mathscr{H}(\mathscr{A}, J)$, $\xi_1, \ldots, \xi_m$ indeterminates, and let $P = \Phi(\xi_1, \ldots, \xi_m)$. We have the generic element

$$x = \sum_1^m \xi_i u_i \tag{4.1}$$

in $\mathscr{H}(\mathscr{A}, J)^P$ whose minimum polynomial can be written as

$$\begin{aligned} m_x(\lambda) = \lambda^r - \tau_1(\xi_1, \ldots, \xi_m)\, \lambda^{r-1} + \cdots \\ + (-1)^r\, \tau_r(\xi_1, \ldots, \xi_m), \end{aligned} \tag{4.2}$$

where τ_i is a homogeneous polynomial of degree i in the ξ's. Here $m = n(n+1)/2$ or $n(n-1)/2$ and $r = n$ or $n/2$ according as J is of orthogonal or symplectic type ($n = \deg \mathscr{A}$). We have

$$N_J(x) = \tau_r(\xi_1, ..., \xi_m) \tag{4.3}$$

and

$$m_x(\lambda) = N_J(\lambda 1 - x). \tag{4.4}$$

Let $(v_1, ..., v_m)$ be a second base for $\mathscr{H}(\mathscr{A}, J)$, $v_i = \sum_{j=1}^{m} \alpha_{ji} u_j$ and put $y = \sum \xi_i v_i = \sum \xi_1' u_j$, where $\xi_j' = \sum \alpha_{ji} \xi_i$. Then $N_J(y) = \tau_r(\xi_1', ..., \xi_m')$ and $m_y(\lambda) = N_J(\lambda 1 - y) = \lambda^r - \tau_1(\xi_1', ..., \xi_m')\, \lambda^{r-1} + \cdots + (-1)^r \tau_r(\xi_1', ..., \xi_m')$. Hence $m_y(\lambda)$ and $N_J(y)$ are obtained by making a linear change of variables in $m_x(\lambda)$ and $N_J(x)$, respectively. We recall also that if u is an invertible element of $\mathscr{H}(\mathscr{A}, J)$ and $J_u = i_u J$, then $\mathscr{H}(\mathscr{A}, J_u) = u\mathscr{H}(\mathscr{A}, J)$. Hence $(uu_1, ..., uu_n)$ is a base for $\mathscr{H}(\mathscr{A}, J_u)$ and $N_J(\sum \xi_i uu_i)$ is a multiple of $N_J(x)$.

The polynomial $N_J(x)$ is irreducible in $\Phi[\xi_1, ..., \xi_m]$ ([6, Theorem 7]). This implies that $m_x(\lambda)$ is irreducible in $\Phi[\xi_1, ..., \xi_m, \lambda]$ and hence in $\Phi(\xi_1, ..., \xi_m)[\lambda]$. Also the discriminant of $m_x(\lambda)$ as a polynomial in λ is $\neq 0$. It follows that

$$\Phi(\xi_1, ..., \xi_m, x) = \Phi(\xi_1, ..., \xi_m)[x] \tag{4.5}$$

is a separable algebraic extension of $\Phi(\xi_1, ..., \xi_m)$ and hence $\Phi(\xi_1, ..., \xi_m, x)$ is separable over Φ.

Since $N_J(x) = \tau_r(\xi_1, ..., \xi_m)$ is irreducible in $\Phi[\xi_1, ..., \xi_m]$,

$$\Phi[\xi_1, ..., \xi_m]/(\tau_r(\xi_1, ..., \xi_m)) \tag{4.6}$$

is a domain. Now we have

$$\Phi[\xi_1, ..., \xi_m]/(\tau_r(\xi_1, ..., \xi_m)) = \Phi[\bar{\xi}_1, ..., \bar{\xi}_m],$$

where $\bar{\xi}_i = \xi_i + (\tau_r(\xi_1, ..., \xi_m))$ and we can form the field of fractions $\Phi(\bar{\xi}_1, ..., \bar{\xi}_m)$ which we shall call a *generic norm field* of the Jordan algebra $\mathscr{H}(\mathscr{A}, J)$. It is readily seen from the remarks we made before that generic norm fields determined by different bases for $\mathscr{H}(\mathscr{A}, J)$ and by different involutions of the same type are isomorphic. Now we have the following analogue of a result of Heuser ([5, Satz 2]):

4.7 PROPOSITION. *The field $\Phi(\xi_1, ..., \xi_m, x)$ is isomorphic to a simple transcendental extension of $\Phi(\bar{\xi}_1, ..., \bar{\xi}_m)$.*

Proof. Let $1 = \sum \alpha_i u_i$, $\alpha_i \in \Phi$, and put

$$y_i = \alpha_i x - \xi_i 1, \qquad 1 \leqslant i \leqslant m. \tag{4.8}$$

Then $\Phi(\xi_1,..., \xi_m, x) = \Phi(y_1,..., y_m, x)$ and

$$\begin{aligned} m_x(\lambda) = N_J(\lambda 1 - x) = N_J\left(\sum (\alpha_i \lambda - \xi_i)\, u_i\right) \\ = \tau_r(\alpha_1 \lambda - \xi_1,..., \alpha_m \lambda - \xi_m). \end{aligned} \tag{4.9}$$

Since $m_x(x) = 0$,

$$\tau_r(y_1,..., y_m) = 0. \tag{4.10}$$

Hence $\operatorname{tr} \deg_\Phi \Phi(y_1,..., y_m) \leqslant m - 1$ and $\operatorname{tr} \deg_\Phi \Phi(\xi_1,..., \xi_m, x) = \operatorname{tr} \deg_\Phi \Phi(y_1,..., y_m, x) \leqslant m$. Since the ξ_i are algebraically independent, $\operatorname{tr} \deg_\Phi \Phi(\xi_1,..., \xi_m, x) = m$. It follows that $\operatorname{tr} \deg_\Phi \Phi(y_1,..., y_m) = m - 1$ and $\Phi(y_1,..., y_m, x)$ is a simple transcendental extension of $\Phi(y_1,..., y_m)$. Now consider the homomorphism ν of $\Phi[\eta_1,..., \eta_m]/\Phi$, η_i indeterminates, onto $\Phi[y_1,..., y_m]$ such that $\eta_i \rightsquigarrow y_i$, $1 \leqslant i \leqslant m$. By (4.10), $\ker \nu$ contains the prime ideal $(\tau_r(\eta_1,..., \eta_m))$. Since $\operatorname{tr} \deg_\Phi \Phi(y_1,..., y_m) = m - 1$, $\ker \nu = (\tau_r(\eta_1,..., \eta_m))$ (see, e.g., [17, p. 193]). Hence

$$\Phi[y_1,..., y_m] \cong \Phi[\eta_1,..., \eta_m]/(\tau_r(\eta_1,..., \eta_r))$$

and $\Phi(y_1,..., y_m) \cong \Phi(\bar{\xi}_1,..., \bar{\xi}_m)$. Thus $\Phi(\xi_1,..., \xi_m, x)$ is isomorphic to a simple transcendental extension of $\Phi(\bar{\xi}_1,..., \bar{\xi}_m)$. ∎

If E is an extension field of Φ, then $(u_1,..., u_m)$ is a base for $\mathscr{A}^E$. The minimum polynomial and generic norm of $x = \sum \xi_i u_i$ are unchanged on passing from $\mathscr{A}$ to $\mathscr{A}^E$. Hence $\tau_r(\xi_1,..., \xi_m)$ is irreducible for every extension field E/Φ, which means that $N_J(x) = \tau_r(\xi_1,..., \xi_m)$ is absolutely irreducible. The generic norm field determined by x (for $((\mathscr{H}, J)^E)$ is the field of fractions of $E[\xi_1,..., \xi_m]/(\tau_r(\xi_1,..., \xi_m)) \cong E \otimes \Phi[\xi_1,..., \xi_m]/(\tau_r(\xi_1,..., \xi_m))$. Hence $\Phi(\bar{\xi}_1,..., \bar{\xi}_m)$ is isomorphic to a subfield of $E(\bar{\xi}_1,..., \bar{\xi}_m)$. If $[E: \Phi] < \infty$, then $E \otimes_\Phi \Phi(\bar{\xi}_1,..., \bar{\xi}_m)$ is the field of fractions of $E[\xi_1,..., \xi_m]/(\tau_r(\xi_1,..., \xi_m))$. This implies that $\Phi(\bar{\xi}_1,..., \bar{\xi}_m)/\Phi$ contains no finite-dimensional subfield and hence Φ is algebraically closed in $\Phi(\bar{\xi}_1,..., \bar{\xi}_m)$. We observe also that since $\Phi(\xi_1,..., \xi_m, x)$ is separable over Φ, $\Phi(\bar{\xi}_1,..., \bar{\xi}_m)$ which is isomorphic to a subfield of $\Phi(\xi_1,..., \xi_m, x)/\Phi$ is separable. Now an extension field E/Φ is called a regular extension of Φ in the sense of Weil if E/Φ is separable and Φ is algebraically closed in E. Hence we have

4.11. THEOREM. *Any generic norm field $\Phi(\bar{\xi}_1,..., \bar{\xi}_m)$ is a regular extension of Φ.*

We recall the definition of a Φ-place from an extension field E/Φ to an extension field E'/Φ: a homomorphism $\mathscr{P}$ of a Φ-subalgebra R into E'/Φ such that if $a \in E$, $\notin R$, then $a^{-1} \in R$ and $\mathscr{P}(a^{-1}) = 0$. We also have

4.12. DEFINITION. Let E_i, $i = 1, 2$, be an extension field of Φ. Then E_1 and E_2 are *Φ-place equivalent* if there exists a Φ-place from E_1/Φ to E_2/Φ and a Φ-place from E_2/Φ to E_1/Φ.

The following result is readily established by induction on m using the fact that the composite of Φ-places is a Φ-place:

4.13. PROPOSITION. *If the ξ_i, $1 \leqslant i \leqslant m$, are indeterminates, then Φ and $\Phi(\xi_1, \ldots, \xi_m)$ are Φ-place equivalent.*

We omit the proof.

We also have the following result [4, p. 34; 11, p. 428]:

4.14. PROPOSITION. *Let E/Φ be a splitting field for a central simple algebra $\mathscr{A}/\Phi$ and suppose there is a Φ-place from E/Φ to E'/Φ. Then E'/Φ is a splitting field for $\mathscr{A}$.*

We can now prove the following result on involutions of orthogonal type:

4.15. PROPOSITION. *If J is of orthogonal type, then any generic norm field $\Phi(\bar{\xi}_1, \ldots, \bar{\xi}_m)$ determined by $N_J(x)$ is a splitting field for $\mathscr{A}$.*

Proof. If $x = \sum \xi_i u_i$, then $m_x(\lambda)$ is of degree $r = n$. Hence $\Phi(\xi_1, \ldots, \xi_m, x)$ is a maximal subfied of $\mathscr{A}^P$, $P = \Phi(\xi_1, \ldots, \xi_m)$. Hence this is a splitting field for $\mathscr{A}^P$ and for $\mathscr{A}$. Since $\Phi(\xi_1, \ldots, \xi_m, x)$ is a simple transcendental extension of $\Phi(\bar{\xi}_1, \ldots, \bar{\xi}_m)$, we have a Φ-place from $\Phi(\xi_1, \ldots, \xi_m, x)$ to $\Phi(\bar{\xi}_1, \ldots, \bar{\xi}_m)$. Hence, by 4.14, $\Phi(\bar{\xi}_1, \ldots, \bar{\xi}_m)$ is a splitting field for $\mathscr{A}$. ∎

We shall require also the following well-known result:

4.16. PROPOSITION. (i) *If $\mathscr{D}$ is a finite-dimensional division algebra over Φ and $\xi_1, \ldots, \xi_m$ are indeterminates, then $\mathscr{D}^{\Phi(\xi_1, \ldots, \xi_m)}$ is a division algebra.*

(ii) *The canonical map $\{\mathscr{A}\} \rightsquigarrow \{\mathscr{A}^{\Phi(\xi_1, \ldots, \xi_m)}\}$ of* Br Φ *into* Br$(\Phi(\xi_1, \ldots, \xi_m))\}$ *is injective.*

Proof. (i) The algebra $\mathscr{D}^{\Phi(\xi_1, \ldots, \xi_m)}$ contains the subalgebra $\mathscr{D}^{\Phi[\xi_1, \ldots, \xi_m]} \cong \mathscr{D}[\xi_1, \ldots, \xi_m]$ and this is a domain. Every element of $\mathscr{D}^{\Phi(\xi_1, \ldots, \xi_m)}$ can be written in th form $\varphi(\xi_1, \ldots, \xi_m)^{-1} f$ when $\varphi \in \Phi[\xi_1, \ldots, \xi_m]$ and $f \in \mathscr{D}^{\Phi[\xi_1, \ldots, \xi_m]}$. It follows that $\mathscr{D}^{\Phi(\xi_1, \ldots, \xi_m)}$ is a domain and since it is finite dimensional over $\Phi(\xi_1, \ldots, \xi_m)$, it is a division algebra.

(ii) This is an immediate consequence of (i). ∎

Our first main result on generic norm fields of $\mathscr{H}(\mathscr{A}, J)$ is

4.17. THEOREM. *If J is of orthogonal type, then $\mathscr{A} \sim 1$ if and only if the generic norm field $\Phi(\bar{\xi}_1,\dots,\bar{\xi}_m)$ of N_J is a rational function field.*

Proof. Suppose first that $\mathscr{A} \sim 1$ so $\mathscr{A} = M_n(\Phi)$. We may assume $J = t$ the transpose involution and the base for $\mathscr{H}(\mathscr{A}, J)$ is $\{e_{ii}, e_{ij} + e_{ji}, i < j = 1,\dots, n\}$. Then $x = \sum \xi_{ii} e_{ii} + \sum_{i<j} \xi_{ij}(e_{ij} + e_{ji})$ with ξ_{ii}, ξ_{ij} indeterminates. Then $N_J(x) = \det x$, which is of first degree in ξ_{ii}. It follows that $\bar{\xi}_{ii}$, $i > 1$, $\bar{\xi}_{ij}$, $i < j$, is a transcendency base for the generic norm field $\Phi(\bar{\xi}_{ii}, \bar{\xi}_{ij})$. Conversely, assume $\Phi(\bar{\xi}_1,\dots,\bar{\xi}_m)$ is rational. By Proposition 4.15, $\Phi(\bar{\xi}_1,\dots,\bar{\xi}_m)$ is a splitting field for $\mathscr{A}$. Hence, by Proposition 4.16, $\mathscr{A} \sim 1$. ∎

4.18. DEFINITION (Amitsur). An extension field E/Φ is a *generic splitting field* for a central simple algebra $\mathscr{A}/\Phi$ if it has the property that an extension field E'/Φ is a splitting field for $\mathscr{A}$ if and only if there exists a Φ-place from E/Φ to E'/Φ.

Since there exists a Φ-place from E/Φ to E/Φ, the definition implies that any generic splitting field is a splitting field.

We can now prove the main theorem on generic norm field for involutions of orthogonal type.

4.19. THEOREM. *Let $\mathscr{A}$ be an involutorial central simple algebra over Φ, J an involution of orthogonal type in $\mathscr{A}$. Then any generic norm field determined by an $N_J(x)$ is a generic splitting field for $\mathscr{A}/\Phi$.*

Proof. Let $\Phi(\bar{\xi}_1,\dots,\bar{\xi}_m)$ be the generic norm field determined by $x = \sum \xi_i u_i$, where $(u_1,\dots,u_m)$ is a base for $\mathscr{H}(\mathscr{A}, J)$. We have seen that $\Phi(\bar{\xi}_1,\dots,\bar{\xi}_m)$ is a splitting field for $\mathscr{A}$. Hence if there exists a Φ-place from $\Phi(\bar{\xi}_1,\dots,\bar{\xi}_m)$ to E/Φ, then E is a splitting field for $\mathscr{A}$. Conversely, suppose E is a splitting field for $\mathscr{A}$. Then $\mathscr{A}^E = M_n(E)$ and hence the generic norm field $E(\bar{\xi}_1,\dots,\bar{\xi}_m)$ of $\mathscr{A}^E$ is rational. Then there exists an E-place from $E(\bar{\xi}_1,\dots,\bar{\xi}_m)$ to E. On the other hand, $\Phi(\bar{\xi}_1,\dots,\bar{\xi}_m)$ is isomorphic to a subfield of $E(\bar{\xi}_1,\dots,\bar{\xi}_m)$. If we identify $\Phi(\bar{\xi}_1,\dots,\bar{\xi}_m)$ with the corresponding subfield of $E(\bar{\xi}_1,\dots,\bar{\xi}_m)$, then the restriction of $\mathscr{P}$ to $\Phi(\bar{\xi}_1,\dots,\bar{\xi}_m)$ is a Φ-place from $\Phi(\bar{\xi}_1,\dots,\bar{\xi}_m)$ to E. ∎

We now consider involutions J of symplectic type. Let $\Phi(\bar{\xi}_1,\dots,\bar{\xi}_m)$ be a generic norm field determined by such an involution. We shall show that $\Phi(\bar{\xi}_1,\dots,\bar{\xi}_m)$ is a $\frac{1}{2}$-splitting field for $\mathscr{A}$ in the sense of the following

4.20. DEFINITION. If $\mathscr{A}$ is central simple over Φ, an extension field E/Φ is a *$(1/s)$-splitting field for $\mathscr{A}$* if $\mathscr{A}^E$ has index $\leqslant s$ or, equivalently, $\mathscr{A}^E$ has a splitting field of degree $\leqslant s$.

If $s \mid n$ and E is a subfield of $\mathscr{A}$ such that $[E\colon \Phi] = n/s$, then $\mathscr{A}^E$ has the same index as the centralizer $C_{\mathscr{A}}(E)$, which is central simple over E. Moreover, $[C_{\mathscr{A}}(E)\colon \Phi][E\colon \Phi] = [\mathscr{A}\colon \Phi] = n^2$, which implies that $[C_{\mathscr{A}}(E)\colon E] = s^2$. Then the index of $C_{\mathscr{A}}(E) \leqslant s$. Thus if $\mathscr{A}$ has a subfield of degree $\leqslant n/s$, then E is a $(1/s)$-splitting field for $\mathscr{A}$.

4.21. Proposition. *If $\mathscr{A}$ is involutorial and J is an involution of symplectic type in $\mathscr{A}$, then the generic norm field $\Phi(\bar{\xi}_1,\dots,\bar{\xi}_m)$ determined by N_J is a $\frac{1}{2}$-splitting field for $\mathscr{A}$.*

Proof. The field $\Phi(\bar{\xi}_1,\dots,\bar{\xi}_m)$ is a subfield of degree $r = n/2$ of $\mathscr{A}^{\Phi(\xi_1,\dots,\xi_m)}$ and for $\mathscr{A}$. Now $\Phi(\xi_1,\dots,\xi_m, x)$ is a simple transcendental extension of $\Phi(\bar{\xi}_1,\dots,\bar{\xi}_m)$. Hence, by Proposition 4.16, $\mathscr{A},\dots, \mathscr{A}^{\Phi(\xi_1,\dots,\xi_m,x)}$ and $\mathscr{A}^{\Phi(\bar{\xi}_1,\dots,\bar{\xi}_m)}$ have the same index $\leqslant 2$. Then $\Phi(\bar{\xi}_1,\dots,\bar{\xi}_m)$ is a $\frac{1}{2}$-splitting for $\mathscr{A}$.

We prove next an analogue for involutions of symplectic type of Theorem 4.17.

4.22. Theorem. *Let $\mathscr{A}$ be involutorial, J an involution of symplectic type, and let $\Phi(\bar{\xi}_1,\dots,\bar{\xi}_m)$ be a generic norm field determined by N_J. Then the index of $\mathscr{A} \leqslant 2$ if and only if $\Phi(\bar{\xi}_1,\dots,\bar{\xi}_m)$ is rational.*

Proof. Suppose first that $\mathscr{A} = M_r(\mathscr{D})$, where $\mathscr{D}$ is a quaternion algebra (possibly split) and J is of symplectic type. To encompass the case char $\Phi = 2$ we assume $\mathscr{D}$ is generated by u and v such that

$$u^2 = u + \alpha, \qquad vu = (1-u)v, \qquad v^2 = \beta, \tag{4.23}$$

where $\alpha, \beta \in \Phi$, $4\alpha^2 + 1 \neq 0$, $\beta \neq 0$. Here $\Phi[u]$ has the automorphism $a \rightsquigarrow \bar{a}$, where $\bar{u} = 1 - u$, which extends to the involution in $\mathscr{D}$ such that $\bar{v} = -v$. This is the standard involution $\mathscr{D}$. We have a representation of $\mathscr{D}$ in $M_2(\Phi[v])$ such that if $a, b \in \Phi[u]$, then

$$d = a + bv \rightsquigarrow \begin{pmatrix} a & b \\ \beta\bar{b} & \bar{a} \end{pmatrix}. \tag{4.24}$$

We have

$$\bar{d} = \bar{a} - bv \rightsquigarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ({}^t d) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Now consider $\mathscr{A} = M_r(\mathscr{D})$. This has the involution $\bar{t}\colon (d_{ij}) \rightsquigarrow {}^t(\bar{d}_{ij})$. Since $[\mathscr{H}(\mathscr{A}, \bar{t})\colon \Phi] = r(2r-1)$ and $\deg \mathscr{A} = 2r$, $\bar{t}$ is of symplectic type. We have the representation of $M_r(\mathscr{D})$ in $M_{2r}(\Phi[u])$ such that if $d_{ij} = a_{ij} + b_{ij}v$, where $a_{ij}, b_{ij} \in \Phi[u]$, then

$$(d_{ij}) \rightsquigarrow \left(\begin{pmatrix} a_{ij} & b_{ij} \\ \beta\bar{b}_{ij} & \bar{a}_{ij} \end{pmatrix}\right). \tag{4.25}$$

Now $(d_{ij}) \in \mathscr{H}(\mathscr{A}, \bar{t})$ if and only if $\bar{d}_{ii} = d_{ii} = \alpha_{ii} \in \Phi$ and $d_{ji} = \bar{d}_{ij}$ for $i < j$. The corresponding matrix in $M_{2r}(\Phi[u])$ has the form

$$A = \begin{bmatrix} \alpha_{11} & 0 & a_{12} & b_{12} & \cdots \\ 0 & \alpha_{11} & \beta\bar{b}_{12} & \bar{a}_{12} & \cdots \\ \bar{a}_{12} & -b_{12} & \alpha_{22} & 0 & \cdots \\ -\beta\bar{b}_{12} & a_{12} & 0 & \alpha_{22} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}. \tag{4.26}$$

The condition defining this matrix is $s({}^tA)s^{-1} = A$, where

$$s = \operatorname{diag}\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\dots,\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}. \tag{4.27a}$$

This is equivalent to As, is alternate, and hence $N_{\bar{t}}(A) = \mathrm{Pf}(As)$. Now

$$As = \begin{bmatrix} 0 & \alpha_{11} & -b_{12} & a_{12} & \cdots \\ -\alpha_{11} & 0 & -\bar{a}_{12} & \beta\bar{b}_{12} & \cdots \\ b_{12} & \bar{a}_{12} & 0 & \alpha_{22} & \cdots \\ -a_{12} & -\beta\bar{b}_{12} & -\alpha_{22} & 0 & \cdots \\ \vdots & \vdots & \vdots & & \end{bmatrix}. \tag{4.27b}$$

It follows that $N_t(A) = \mathrm{Pf}(As)$ is linear in α_{11}. Hence if we choose as base for $\mathscr{H}(\mathscr{A}, \bar{t})$: $\{e_{ii}, d_k e_{ij} + \bar{d}_k e_{ji}\}$, where $i < j$, and (d_1, d_2, d_3, d_4) is a base for $\mathscr{D}$, then $N_{\bar{t}}(x)$ for $x = \sum \xi_{ii} e_{ii} + \sum_{i<j} \xi_{kij}\,(d_k e_{ij} + \bar{d}_k e_{ji})$, the ξ indeterminates, then the $\bar{\xi}_{ii}$, $i > 1$, $\bar{\xi}_{kij}$, are algebraically independent and $\Phi(\bar{\xi}_{ii}, i > 1, \bar{\xi}_{kij})$ is a generic norm field for $\mathscr{H}(\mathscr{A}, J)$. Hence this field is rational.

Conversely, suppose the generic norm field $\Phi(\bar{\xi}_1,\dots, \bar{\xi}_m)$ determined by N_J is rational. Since $\Phi(\bar{\xi}_1,\dots, \bar{\xi}_m)$ is a $\frac{1}{2}$-splitting field, $\mathscr{A}^{\Phi(\bar{\xi}_1,\dots,\bar{\xi}_m)} = M_r(\bar{\mathscr{D}})$, where $\bar{\mathscr{D}}$ is a quaternion algebra over $\Phi(\bar{\xi}_1,\dots, \bar{\xi}_m)$. It follows from Proposition 4.16 that the index of $\mathscr{A} \leqslant 2$. ∎

4.28. Definition. An extension field E/Φ is a *generic* $(1/s)$*-splitting field* for a central simple algebra $\mathscr{A}/\Phi$ if it has the following property: An extension field E'/Φ is a $(1/s)$-splitting field if and only if there exists a Φ-place from E/Φ to E'/Φ.

We can now prove our main result for generic norm fields determined by involutions of symplectic type.

4.29. THEOREM. *If J is of symplectic type, then any generic norm field determined by an $N_J(x)$ is a generic $\frac{1}{2}$-splitting field for $\mathscr{A}$.*

Proof. If $\Phi(\bar{\xi}_1,..., \bar{\xi}_m)$ is a generic norm field determined by $N_J(x)$, then $\mathscr{A}^{\Phi(\bar{\xi}_1,...,\bar{\xi}_m)} = M_r(\mathscr{D})$, where $\mathscr{D}$ is a quaternion algebra. It is readily seen that $\mathscr{D}$ is a division algebra. Hence if $\bar{\xi}_{m+1} \in \mathscr{D}$, $\notin \Phi(\bar{\xi}_1,..., \bar{\xi}_m)$, then $\Phi(\bar{\xi}_1,..., \bar{\xi}_{m+1})$ is a splitting field for $\mathscr{A}$. Now let E be an extension field of Φ such that there exists a Φ-place $\mathscr{P}$ from $\Phi(\bar{\xi}_1,..., \bar{\xi}_m)$ to E. Then $\mathscr{P}$ can be extended to a Φ-place $\mathscr{P}'$ from $\Phi(\bar{\xi}_1,..., \bar{\xi}_{m+1})$ into algebraic extension E' of E. Let 0, P ($0'$, P') be the valuation ring and maximal ideal associated with $\mathscr{P}(\mathscr{P}')$. Then $0'/P'$ is an extension field of $0/P$ and since $[\Phi(\bar{\xi}_1,..., \bar{\xi}_{m+1}): \Phi(\bar{\xi}_1,..., \bar{\xi}_m)] = 2$, $[0'/P' : 0/P] \leqslant 2$. Now $0/P$ is isomorphic to a subfield of E. Hence we have a Φ-place from $\Phi(\bar{\xi}_1,..., \bar{\xi}_{m+1})$ into either E or a quadratic field extension E' of E. Thus either E or E' is a splitting field for $\mathscr{A}$ and E is a $\frac{1}{2}$-splitting field for $\mathscr{A}$.

Conversely assume that E is a $\frac{1}{2}$-splitting field for $\mathscr{A}$. Then the argument used in the proof of Theorem 4.19 shows that there exists a Φ-place from $\Phi(\bar{\xi}_1,..., \bar{\xi}_{m+1})$ to E. ∎

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